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SOME RESULTS ON FIXED POINT THEORY IN CONE METRIC SPACES

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Abstract

We prove two fixed point theorems for mappings satisfying integral type contractive condition. These results extend the work of B.E.Rhoades [7] and Hardy and Rogers [8] to cone metric spaces.

1. Introduction and Preliminaries

In 2007, Huang and Zhang [3] introduced the notion of cones and defined cone metric spaces by replacing the real numbers by an ordered Banach space, wherein they established the convergence of sequences and its completeness and proved some fixed point theorems for mappings satisfying certain contractive conditions in the framework of normal cone metric spaces. Subsequently, many fixed point theorems have been proved by several authors [4-7] in the setting of cone metric spaces in which the cone does not need to be normal.

Recently, Farshid Khojasteh, Zahra Goodarzi and Abdolrahman Razani [2], defined a concept of integral with respect to a cone. The results in [2] are the extension of Branciari's work [1].

The aim of the paper is to prove fixed point theorems for mappings satisfying a general contractive condition of integral type in the setting of normal cone metric spaces. In order to do this the following definitions and results will be needed.

Definition 1.1.

Let E always be a real Banach space and P a subset of E.Then P is called a cone in E if and only if:

(i) P is closed, non-empty and $P \neq \{0\}$,

(ii) $a, b \in R$, $a, b \geq 0$, $x, y \in P$ implies $ax + by \in P$,

(iii) $x \in P$ and $-x \in P$ implies $x = 0$ that is $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, a partial ordering $\leq$ on E with respect to P is defined by $x \leq y$ if and only if $y - x \in P$. We shall denote $x < y$ to indicate $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int } P$, where int P denotes the interior of P.

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Definition 1.2.

A cone $P \subset E$ is called normal if

$$\inf\{\ \|x + y\| : x, y \in P, \|x\| = \|y\| = 1\} > 0$$

or, equivalently, if there is a number $M > 0$ such that for all $x, y \in P$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq M\|y\|. \tag{1.1}$$

the least positive number satisfying (1.1) is called the normal constant of P.It is clear that $M \geq 1$. A cone $P \subset E$ is nonnormal if and only if there exist sequences $\{x_n\}, \{y_n\} \subset P$ such that $0 \leq x_n \leq x_n + y_n$, $x_n + y_n \to 0$ but $x_n \nrightarrow 0 (n \to \infty)$.

Example 1.3.

Let $E = C_R^1 [0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and consider the cone $P = \{x \in E: x(t) \geq 0 \text{ on } [0,1]\}$. This cone is not normal. For instant, set

$$x_n(t) = \frac{1-\sin nt}{n+2}, \quad y_n(t) = \frac{1+\sin nt}{n+2}.$$

Then, $\|x_n\| = \|y_n\| = 1$ and $\|x_n + y_n\| = 2/(n+2) \to 0$ as $n \to \infty$.

In the sequel, we always suppose that E is a real Banch space, P is a cone in E with $\text{int } P \neq \Phi$ and $\leq$ is a partial ordering with respect to P.

Definition 1.4.

Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(i) $0 \leq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and the pair (X, d) is called a cone metric space.

Example1.5.

Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subseteq R^2$.

Choose $X = R$ and define $d : X \times X \to E$ such that

$$d(x, y) = (\,|x-y|, \alpha\,|x-y|),\ \alpha \geq 0 \text{ is a constant.}$$

Then (X, d) is a cone metric space.

Moreover, the category of cone metric spaces is bigger than the category of metric spaces.

Definition 1.6.

Let (X,d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$, there is n_0 such that for all $n > n_0$, $d(x_n, x) \ll c$, then $\{x_n)$ is said to be convergent and $\{x_n\}$ converges to x.

Lemma 1.7.

Let (X, d) be a cone metric space and P be a normal cone with normal constant M. Let $\{x_n\}_{n \geq 1}$ in X converges to x if and only if

$$d(x_n, x) \rightarrow 0 \text{ as } (n \rightarrow \infty).$$

Definition 1.8

Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X. If for every $c \in E$ with $0 \ll c$, there is n_0 such that for all $n, m > n_0,$

$$d(x_n, x_m) \ll c.$$

Then $\{x_n\}$ is called a Cauchy sequence in X.

Definition 1.9.

Let (X, d) be a cone metric space. If every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Lemma 1.10

Let (X,d) be a cone metric space and P a normal cone with normal constant M. If a sequence $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequency in X.

Lemma 1.11.

Let (X,d) be a cone metric space and P a normal cone with normal constant M. Let $\{x_n\}_{n\geq 1}$ is a Cauchy sequence if and only if $d(x_n,x_m) \rightarrow 0 (n,m \rightarrow \infty)$

Lemma 1.12

Let (X,d) be a cone metric space and P a normal cone with normal constant M. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x_n \rightarrow x$, $y_n \rightarrow y$ then $d(x_n,y_n) \rightarrow d(x,y)$ $(n \rightarrow \infty)$

Theorem 1.13.

Let (X, d) be a complete metric space, $\alpha \in (0, 1)$ and $f: X \to X$ is a mapping such that for all x, $y \in X$, $\int_0^{d(fx,fy)} \varphi(t)dt \leq \alpha \int_0^{d(x,y)} \varphi(t)dt$, where $\varphi: [0, +\infty) \to [0, +\infty)$ is non-negative and summable on each compact subset of $[0, +\infty)$ such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$, then f has a unique fixed point $a \in X$, such that for each $x \in X$, $\lim_{n\to\infty} f^n x = a$.

2. Main Result

Theorem 2.1.

Let (X, d) be a complete cone metric space and P be a normal cone. Suppose that $\varphi : P \to P$ is a non vanishing mapping and a sub additive cone integrable on each $[a,b] \subset P$ such that

$$\text{for each } \varepsilon \gg 0, \int_0^{\varepsilon} \varphi \, d_p \gg 0 \qquad (1)$$

If $f: X \to X$ is a mapping such that for all $x, y \in X$, $\int_0^{d(fx,fy)} \varphi \, d_p \leq \alpha \int_0^{u(x,y)} \varphi \, d_p$, where $u(x,y) = \max\{ d(x, y), d(x, fx), d(y, fy), \frac{d(x,fy)}{2}, \frac{d(fx,y)}{2} \}$ for some $\alpha \in (0,1)$. Then f has a unique fixed point in X.

Proof:

Let $x_1 \in P$. Choose $x_{n+1} = f(x_n)$. Then

$\int_0^{d(x_n,x_{n+1})} \varphi \, d_p = \int_0^{d(fx_{n-1},fx_n)} \varphi \, d_p \leq \alpha \int_0^{u(x_{n-1},x_n)} \varphi \, d_p$,where

$$u(x_{n-1},x_n) = \max\{ d(x_{n-1},x_n), d(x_{n-1},fx_{n-1}), d(x_n,fx_n), \frac{d(x_{n-1},fx_n)}{2}, \frac{d(fx_{n-1},x_n)}{2}\}$$

$$= \max\{ d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{d(x_{n-1},x_{n+1})}{2} \}$$

$$= \max\{ d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{d(x_{n-1},x_{n+1})}{2} \}. \qquad (2)$$

But,

$$d(x_{n-1},x_{n+1}) \leq \frac{d(x_{n-1},x_n) + d(x_n,x_{n+1})}{2} \leq \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}$$

Thus $u(x_{n-1},x_n) = \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}$.

From the equation (2)

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$$\int_0^{d(x_n, x_{n+1})} \varphi \, d_p \leq \alpha \int_0^{\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}} \varphi \, d_p$$

$$= \alpha \max\{\int_0^{d(x_{n-1}, x_n)} \varphi \, d_p , \int_0^{d(x_n, x_{n+1})} \varphi \, d_p\}$$

$$= \alpha \int_0^{d(x_{n-1}, x_n)} \varphi \, d_p$$

$$\leq \alpha^2 \int_0^{d(x_{n-2}, x_{n-1})} \varphi \, d_p$$

$$\ldots\ldots$$

$$\leq \alpha^{n-1} \int_0^{d(x_1, x_2)} \varphi \, d_p .$$

Since $\alpha \in (0,1)$, it follows that

$$\lim_{n\to\infty} \int_0^{d(x_n, x_{n+1})} \varphi \, d_p = 0.$$

If $\lim_{n\to\infty} d(x_n, x_{n+1}) \neq 0$, then $\lim_{n\to\infty} \int_0^{d(x_{n+1}, x_n)} \varphi \, d_p \neq 0$,

This is a contradiction.

Hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$

We now show that $\{x_n\}$ is a Cauchy sequence in X. That is, $\lim_{m,n\to\infty} d(fx_m, fx_n) = 0$.

By Triangle inequality, for $n > m$

$$\int_0^{d(fx_n, fx_m)} \varphi \, d_p = \int_0^{d(x_{n+1}, x_{m+1})} \varphi \, d_p$$

$$\leq \int_0^{d(x_{n+1}, x_n) + d(x_n, x_{n-1}) + \cdots + d(x_{m+2}, x_{m+1})} \varphi \, d_p$$

$$\leq \int_0^{d(x_{n+1}, x_n)} \varphi \, d_p + \int_0^{d(x_n, x_{n-1})} \varphi \, d_p + \cdots \int_0^{d(x_{m+2}, x_{m+1})} \varphi \, d_p$$

$$\leq (\alpha^{n-1} + \alpha^{n-2} + \ldots.. + \alpha^m) \int_0^{d(x_1, x_2)} \varphi \, d_p$$

$$= \alpha^m (1 + \alpha + \cdots + \alpha^{n-m-1}) \int_0^{d(x_1, x_2)} \varphi \, d_p$$

$$= \alpha^m \frac{1-\alpha^{n-m}}{1-\alpha} \int_0^{d(x_1, x_2)} \varphi \, d_p$$

$$\leq \frac{\alpha^m}{1-\alpha} \int_0^{d(x_1, x_2)} \varphi \, d_p$$

As $n, m \to \infty$ and $\alpha \in (0, 1)$, we have

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$$\lim_{m,n\to\infty} \int_0^{d(fx_n,fx_m)} \varphi\, d_p = 0.$$

Thus $$\lim_{m,n\to\infty} d(fx_n, fx_m) = 0.$$

This means that $\{x_n\}$is a Cauchy sequence in X and since X is a complete cone metric space,$\{x_n\}$ converges to some $x_0 \in X$.

Finally

$$\int_0^{d(x_{n+1},fx_0)} \varphi\, d_p = \int_0^{d(fx_n,\, fx_0)} \varphi\, d_p$$

$$\leq \alpha \int_0^{u(x_n,x_0)} \varphi\, d_p \text{ , where}$$

$$u(x_n, x_0) = \max\left\{d(x_n, x_0), d(x_n, fx_n), d(x_0, fx_0), \frac{d(x_n,fx_0)}{2}, \frac{d(fx_n, x_0)}{2}\right\}.$$

Thus $\lim_{n\to\infty} \int_0^{d(x_{n+1},fx_0)} \varphi\, d_p = 0$.

That is, $\lim_{n\to\infty} d(x_{n+1}, fx_0) = 0$.

By the uniqueness of the limit of a sequence, $f(x_0) = x_0$.

This means that x_0 is a fixed point of f in X.

Uniqueness:

If x_0, y_0 are any two distinct fixed points of f, then

$$\int_0^{d(x_0,y_0)} \varphi\, d_p = \int_0^{d(fx_0,fy_0)} \varphi\, d_p$$

$$\leq \alpha \int_0^{u(x_0,y_0)} \varphi\, d_p$$

where $$u(x_0, y_0) = \max\left\{d(x_0, y_0), d(x_0, fx_0), d(y_0, fy_0), \frac{d(x_0.f\, y_0)}{2}, \frac{d(fx_0,y_0)}{2}\right\}$$

$$= \max\left\{d(x_0, y_0), \frac{d(x_0,y_0)}{2}, \frac{d(x_0,y_0)}{2}\right\}$$

$$= d(x_0, y_0).$$

Thus $$\int_0^{d(x_0,y_0)} \varphi\, d_p \leq \alpha \int_0^{d(x_0,y_0)} \varphi\, d_p .$$

This implies that $\int_0^{d(x_0,y_0)} \varphi\, d_p = 0$.

from the equation (1),

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$$d(x_{0,}\, y_0) = 0$$

Therefore $x_0 = y_0$.Thus f has a unique fixed point $x_0 \in X$.

Theorem 2.2.

Let (X, d) be a complete cone metric space and P be a normal cone with normal constant M. Suppose that φ: P → P is a non vanishing mapping and a subadditive cone integrable on each [a, b] ⊂ P such that for each $\varepsilon \gg 0$, $\int_0^{\varepsilon} \varphi\, d_p \gg 0$. If f : X →X is a mapping such that for all x, y ∈ X,

$$\int_0^{d(fx,fy)} \varphi\, d_p \le a_1 \int_0^{d(x,y)} \varphi\, d_p + a_2 \int_0^{d(x,fx)} \varphi\, d_p + a_3 \int_0^{d(y,fy)} \varphi\, d_p$$

$$+ a_4 \int_0^{d(x,fy)} \varphi\, d_p + a_5 \int_0^{d(fx,y)} \varphi\, d_p\, ,$$

where $a_{1,}\, a_{2,}\, a_{3,}\, a_{4,}\, a_5 \in (0,1)$ such that $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, then f has a unique fixed point in X.

Proof :

Let $x_1 \in P$. Choose $x_{n+1} = f(x_n)$.

$$\int_0^{d(x_1,\, x_n)} \varphi\, d_p = \int_0^{d(fx_n,\, fx_{n-1})} \varphi\, d_p$$

$$\le a_1 \int_0^{d(x_n,\, x_{n-1})} \varphi\, d_p + a_2 \int_0^{d(x_n, f\,x_n)} \varphi\, d_p + a_3 \int_0^{d(x_{n-1},\, fx_{n-1})} \varphi\, d_p$$

$$+ a_4 \int_0^{d(x_n,\, fx_{n-1})} \varphi\, d_p + a_5 \int_0^{d(fx_n,\, x_{n-1})} \varphi\, d_p$$

$$= a_1 \int_0^{d(x_n,\, x_{n-1})} \varphi\, d_p + a_2 \int_0^{d(x_n,\, x_{n+1})} \varphi\, d_p + a_3 \int_0^{d(x_{n-1},\, x_n)} \varphi\, d_p$$

$$+ a_5 \int_0^{d(x_{n+1},\, x_{n-1})} \varphi\, d_p$$

$$\le (a_1 + a_3) \int_0^{d(x_n,\, x_{n-1})} \varphi\, d_p + a_2 \int_0^{d(x_n,\, x_{n+1})} \varphi\, d_p$$

$$+ a_5 \int_0^{d(x_{n+1},\, x_n)} \varphi\, d_p + a_5 \int_0^{d(x_n,\, x_{n-1})} \varphi\, d_p$$

$$\int_0^{d(x_{n+1},\, x_n)} \varphi\, d_p \le \frac{a_1+a_3+a_5}{1-a_2-a_5} \int_0^{d(x_n,\, x_{n-1})} \varphi\, d_p$$

$$\int_0^{d(x_{n+1},\, x_n)} \varphi\, d_p \le h \int_0^{d(x_n,\, x_{n-1})} \varphi\, d_p, \text{ where } h = \frac{a_1+a_3+a_5}{1-a_2-a_5}$$

$$\le h^2 \int_0^{d(x_{n-1},\, x_{n-2})} \varphi\, d_p$$

...

$$\leq h^{n-1} \int_0^{d(x_2,\, x_1)} \varphi\, d_p\ .$$

For $n > m$, by triangular inequality,

$$\int_0^{d(fx_n,\, fx_m)} \varphi\, d_p \leq \int_0^{d(fx_n,\, fx_{n-1}) + d(fx_{n-1}+x_{n-2}) + \cdots + d(fx_{m-1}, fx_m)} \varphi\, d_p$$

$$\leq \int_0^{d(x_{n+1},\, x_n)} \varphi\, d_p + \int_0^{d(x_n,\, x_{n-1})} \varphi\, d_p + \ldots + \int_0^{d(x_{m+2},\, x_{m+1})} \varphi\, d_p$$

$$\leq (\alpha^{n-1} + \alpha^{n-2} + \ldots + \alpha^m) \int_0^{d(x_2,\, x_1)} \varphi\, d_p$$

$$= \alpha^m (1+\alpha + \cdots . + \alpha^{n-m-1}) \int_0^{d(x_2,\, x_1)} \varphi\, d_p$$

$$= \alpha^m \frac{1-\alpha^{n-m}}{1-\alpha} \int_0^{d(x_2,\, x_1)} \varphi\, d_p$$

$$\leq \frac{\alpha^m}{1-\alpha} \int_0^{d(x_2,\, x_1)} \varphi\, d_p.$$

Since $\alpha < 1$,

$$\lim_{m,n\to\infty} \int_0^{d(fx_n,\, fx_m)} \varphi\, d_p = 0.$$

Then $\lim_{m,n\to\infty} d(fx_n, fx_m) = 0.$

This implies $\{x_n\}$ is a Cauchy sequence in and since X is a complete cone metric space, $\{x_n\}$ converges to some $x_0 \in X$.

Finally,

$$\int_0^{d(x_{n+1},\, fx_0)} \varphi\, d_p = \int_0^{d(fx_n,\, fx_0)} \varphi\, d_p$$

$$\leq a_1 \int_0^{d(x_n,\, x_0)} \varphi\, d_p + a_2 \int_0^{d(x_n,\, fx_n)} \varphi\, d_p + a_3 \int_0^{d(x_0,\, fx_0)} \varphi\, d_p$$

$$+ a_4 \int_0^{d(x_n,\, fx_0)} \varphi\, d_p + a_5 \int_0^{d(fx_n,\, x_0)} \varphi\, d_p$$

$$\leq a_1 \int_0^{d(x_n,\, x_0)} \varphi\, d_p + a_2 \int_0^{d(x_n,\, x_0)} \varphi\, d_p + a_3 \int_0^{d(x_{n+1},\, x_0)} \varphi\, d_p$$

$$+ a_4 \int_0^{d(x_n,\, fx_0)} \varphi\, d_p + a_5 \int_0^{d(x_{n+1},\, x_0)} \varphi\, d_p\ .$$

as $n \to \infty$, we have,

$$\lim_{n\to\infty} \int_0^{d(x_{n+1},\, fx_0)} \varphi\, d_p \leq \lim_{n\to\infty} a_4 \int_0^{d(x_n,\, fx_0)} \varphi\, d_p.$$

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This implies

$$\lim_{n\to\infty} \int_0^{d(x_{n+1},\, fx_0)} \varphi\, d_p = 0.$$

Thus

$$\lim_{n\to\infty} d(x_{n+1}\ fx_0) = 0.$$

From the uniqueness of limit point $f(x_0) = x_{0,}$ which is a fixed point of f.

Uniqueness:

Suppose y_0 is another fixed point of f such that $f\,(y_0) = y_{0.}$

$$\int_0^{d(x_0,\, y_0)} \varphi\, d_p = \int_0^{d(fx_0,\, fy_0)} \varphi\, d_p$$

$$\leq a_1\int_0^{d(x_0,\, y_0)} \varphi\, d_p + a_2\int_0^{d(x_0,\, fx_0)} \varphi\, d_p + a_3 \int_0^{d(y_0,\, fy_0)} \varphi\, d_p$$

$$+ a_4 \int_0^{d(x_0,\, fy_0)} \varphi\, d_p + a_5 \int_0^{d(fx_0,\, y_0)} \varphi\, d_p$$

$$= a_1 \int_0^{d(x_0,\, x_0)} \varphi\, d_p + a_4\int_0^{d(x_0,\, y_0)} \varphi\, d_p + a_5\int_0^{d(x_0,\, y_0)} \varphi d_p$$

$$= (a_1 + a_4 + a_5)\int_0^{d(x_0,\, y_0)} \varphi\, d_p$$

That is,

$$\int_0^{d(x_0,\, y_0)} \varphi\, d_p \quad \leq (a_1 + a_4 + a_5)\int_0^{d(x_0,\, y_0)} \varphi\, d_p$$

Thus, from the equation (1),

$$\int_0^{d(x_0,\, y_0)} \varphi\, d_p = 0$$

Therefore $d(x_0, y_0) = 0$ and hence $x_0 = y_{0.}$Thus, f has a unique fixed point x_0 in X.

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